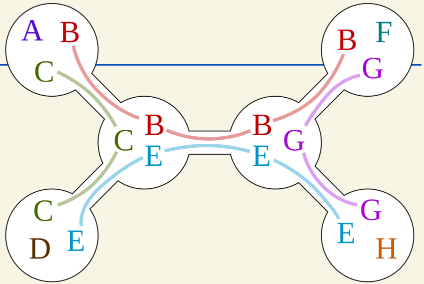
FPT via Linear Programming I



DM898: Parameterized Algorithms
Lars Rohwedder

Today's lecture

Using LP relaxations for FPT algorithms

- LP-based kernel for Vertex Cover
- Next lecture: FPT algorithms for "general" ILPs

LP relaxation of Vertex Cover

ILP formulation

$$\min \sum_{v \in V} x_v$$

$$\begin{aligned} x_u + x_v &\geq 1 && \forall (u,v) \in E \\ x_v &\in \{0,1\} && \forall v \in V \end{aligned}$$

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Lemma

Let x^* be an optimal solution to the LP relaxation of Vertex Cover. There exists a minimum vertex cover S^* with

$$\{v \in V : x_v^* > 1/2\} \subseteq S^* \subseteq V \setminus \{v \in V : x_v^* < 1/2\} \ .$$

Proof: blackboard

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Reduction rule

Solve the LP relaxation and obtain solution x^*. Return $(G - \{v \in V : x^*_v \neq 1/2\}, k - |\{v \in V : x^*_v > 1/2\}|)$.

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LP relaxation of Vertex Cover

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Kernel with $2k$ vertices: Apply previous rule. Then remaining instance has optimal fractional solution $(1/2, 1/2, \ldots, 1/2)$. If $|V| > 2k$ (then optimal solution $\geq$ optimal fractional solution $> k$) return NO.

How much can we trust an LP relaxation?

The previous example shows in particular that for Vertex Cover we can "trust" the LP relaxation on integral variables. Is this a general rule (for all ILPs)?

Unfortunately not

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Consider the following Knapsack instance: There are 5 items of profit 20 and weight 2 and one item of profit 1 and weight 1. ILP formulation:

$$\begin{aligned} \max 20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 + x_6 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 \leq 5 \\ x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{aligned}$$

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The optimal fractional solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 1/2$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$.

The optimal integer solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 1$.

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Not entirely. Fractional solution is still "similar" to integer solution. We will quantify this next lecture

Branch-and-Bound on misguided LP relaxation

This example is also a problem for Branch-and-Bound solvers

$\boxed{A}$

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LP solution:

	x_1	x_2	x_3	x_4	x_5	x_6
A:	1	1	1/2	0	0	0

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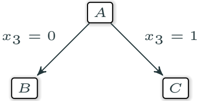

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LP solution:

	x_1	x_2	x_3	x_4	x_5	x_6
A:	1	1	1/2	0	0	0
B:	1	1	0	1/2	0	0
C:	1	1/2	1	0	0	0

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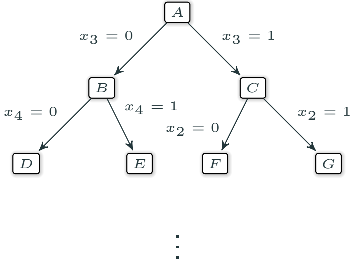

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C:	1	1/2	1	0	0	0
D:	1	1	0	0	1/2	0
E:	1	1/2	0	1	0	0
F:	1	0	0	1/2	0	0
G:	1/2	1	1	0	0	0

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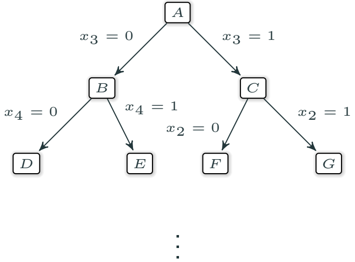

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$\rightsquigarrow$ Branching has almost no effect

Computational experiment

Using different sizes of this example in (naive) Branch-and-Bound solver

$$\begin{aligned} \max 20x_1 + \cdots + 20x_n + x_{n+1} & \\ 2x_1 + \cdots + 2x_n + x_{n+1} &\leq 2k+1 \\ x_1, \ldots, x_{n+1} &\in \{0,1\} \end{aligned}$$

Running time

n	k	time (sec.)	nodes
20	5	3.24	54,262
20	6	6.97	116,278
20	7	12.24	203,400
20	8	17.83	293,928
20	9	21.68	352,714
20	10	21.74	352,714
25	5	13.86	23,228
25	6	38.96	657,798
25	7	92.71	1,562,273
30	5	42.7	736,284
30	6	160.15	2,629,573

Note: Modern ILP solvers would avoid this particular example, e.g., by smart preprocessing, but similar scenarios could still occur