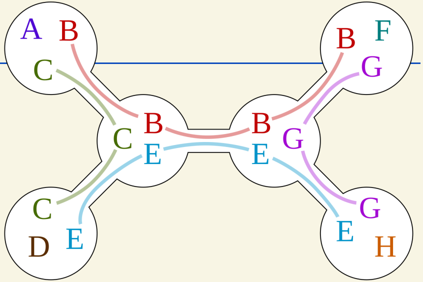
FPT via Linear Programming III



DM898: Parameterized Algorithms
Lars Rohwedder

Today's lecture

- Proof of the proximity theorem
- Steinitz Lemma

Recap of proximity

Problem statement. Given coefficients of the objective $c \in \mathbb{Z}^n$, a matrix $A \in \mathbb{Z}^{m \times n}$ (encoding the coefficients of the constraints in the m rows), right-hand side $b \in \mathbb{Z}^m$ and lower and upper bounds $\ell_i \in \mathbb{Z} \cup \{-\infty\}, u_i \in \mathbb{Z} \cup \{\infty\}$, $i \in \{1, 2, \ldots, n\}$, solve

$$\begin{aligned} \min\ & c^\mathsf{T} x \\ Ax &= b \\ \ell_i \le x_i \le u_i, &\quad x_i \in \mathbb{Z} \qquad \text{for all } i = 1, 2, \ldots, n \end{aligned}$$

Lemma

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\ell_i \in \mathbb{R} \cup \{-\infty\}$, $u_i \in \mathbb{R} \cup \{-\infty\}$ and consider the LP

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Assuming it is feasible and bounded, there is an optimal solution x^* with $|\{i \in \{1, 2, \ldots, n\} : \ell_i < x_i^* < u_i\}| \le m$. Such a solution can be found in polynomial time.

Proximity theorem

Assume that the ILP above is feasible and bounded. Let x^* be an optimal solution to the LP relaxation of the ILP above with at most m non-integral variables. Then there exists an optimal integer solution x with

$$\|x - x^*\|_1 \le (2m^2\Delta + 1)^m + m\ .$$

Few non-tight variables

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- Let x be any optimal solution to the LP (can be computed in polynomial time)
- Let $i_1 < i_2 < \cdots < i_h$ be the indices of variables with $\ell_{i_j} < x_{i_j} < u_{i_j}$

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- Otherwise, $A_{i_1}, \ldots, A_{i_h}$ (columns of A) are **linearly dependent**: There are $\lambda_1, \ldots, \lambda_h \in \mathbb{R}$ not all zero with $\lambda_1 A_{i_1} + \cdots + \lambda_h A_{i_h} = 0$

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Augment x as follows:

$$x'_i = \begin{cases} x_i + \delta\lambda_j & \text{if } i = i_j \\ x_i & \text{if } i \notin \{i_1, \ldots, i_h\} \end{cases}$$

This maintains the validity of constraints and does not decrease the objective.

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This maintains the validity of constraints and does not decrease the objective.

If we choose $\delta > 0$ small enough then no variable bounds will be violated. We can set it carefully so that one additional variable will hit exactly one of its bounds. Repeat the procedure until $h \le m$

Proof of proximity theorem

Preparations

- Let x^* be an optimal solution to the LP relaxation with $\leq m$ non-integer variables
- Let x be an optimal integer solution with $\|x - x^*\|_1$ **minimal** among all optimal integer solutions

x^*	0.5	3	0	6.7	2	0
x	2	2	0	2	2	2

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- Let r be the amount we need to move each variable of x^* to reach the next integer towards x

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- Let r be the amount we need to move each variable of x^* to reach the next integer towards x
- Decompose the remaining change $(x - (x^* + r))$ into vectors $d^{(1)}, \ldots, d^{(h)}$ that each increases or decreases a single variable by 1

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$d^{(1)}$	1	0	0	0	0	0
$d^{(2)}$	0	-1	0	0	0	0
$d^{(3)}$	0	0	0	-1	0	0
$d^{(4)}$	0	0	0	-1	0	0
$d^{(5)}$	0	0	0	-1	0	0
$d^{(6)}$	0	0	0	-1	0	0
$d^{(7)}$	0	0	0	0	0	1
$d^{(8)}$	0	0	0	0	0	1

Note that $x = x^* + r + d^{(1)} + \cdots + d^{(h)}$

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Applying a subset of changes

Consider $z = x^* + \sum_{p \in P} p$ for some $P \subseteq \{r, d^{(1)}, \ldots, d^{(h)}\}$. Then

- z respects the variable bounds, i.e., $\ell_i \leq z_i \leq u_i$
- $Az = b$ if and only if $\sum_{p \in P} Ap = 0$
- z may or may not be integral depending on whether $r \in P$

Lemma

Let $\emptyset \neq P \subsetneq \{r, d^{(1)}, \ldots, d^{(h)}\}$. Then $\sum_{p \in P} Ap \neq 0$.

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Proof. Assume towards contradiction that $\sum_{p \in P} Ap = 0$.

Without loss of generality, assume $r \notin P$. Otherwise, replace P by $P' = \{r, d^{(1)}, \ldots, d^{(h)}\} \setminus P$

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Case 1: $\sum_{p \in P} c^{\mathsf{T}} p < 0$

- Then $z = x^* + \sum_{p \in P} p$ is a feasible fractional solution and $c^{\mathsf{T}} z = c^{\mathsf{T}} x^* + \sum_{p \in P} c^{\mathsf{T}} p < c^{\mathsf{T}} x^*$. Hence, x^* is not optimal. A contradiction

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Case 2: $\sum_{p \in P} c^\mathsf{T} p \geq 0$

- Then $x' = x - \sum_{p \in P} p$ is a feasible integer solution and $c^\mathsf{T} x' = c^\mathsf{T} x - \sum_{p \in P} c^\mathsf{T} p \leq c^\mathsf{T} x$. Hence, x' is also an optimal integer solution. Since we applied some of the changes towards x^* ($P \neq \emptyset$), it is closer to x^* than x. A contradiction

Notice that $Ad^{(1)}, \dots, Ad^{(h)}, Ar$ are all integer vectors with components bounded by $m\Delta$ in absolute value.

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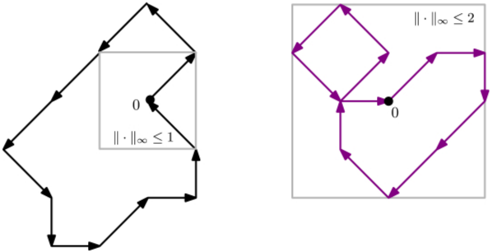
Steinitz Lemma

Let $v_1, \ldots, v_n \in \mathbb{R}^m$ with $v_1 + \cdots + v_n = 0$. There exists a permulation $\sigma \in \mathcal{S}_n$ such that for all $i \in \{1, \ldots, n\}$,

$$\left\|v_{\sigma(1)} + v_{\sigma(2)} + \cdots + v_{\sigma(i)}\right\| \leq m \cdot \max_{j=1}^{n} \|v_j\| \ .$$

Here, $\|\cdot\|$ is an arbitrary norm.



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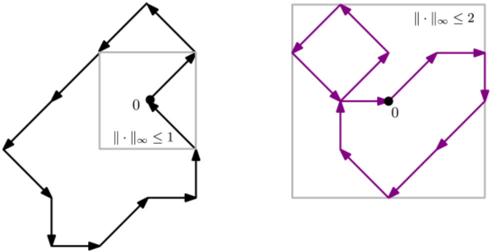

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Proof of proximity theorem (assuming Steinitz Lemma).

- Let $v_1 = Ad^{(1)}, v_2 = Ad^{(2)}, \dots, v_h = Ad^{(h)}, v_{h+1} = Ar$
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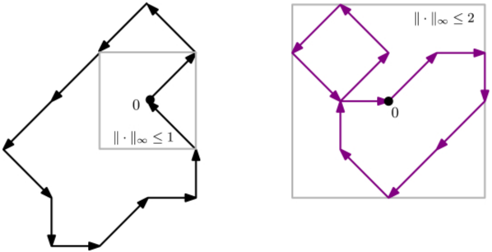

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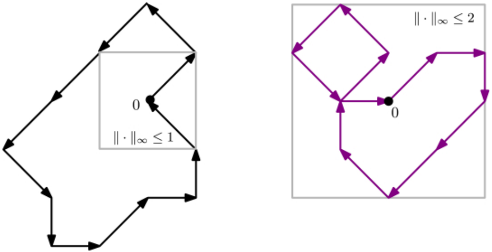

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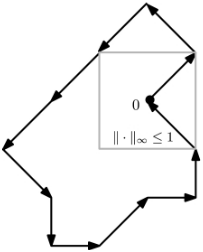

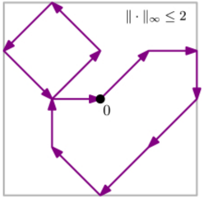

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- Thus $\|x - x^*\|_1 \leq \sum_{i=1}^{h} \|d^{(i)}\|_1 + \|r\|_1 \leq h + m < (2m^2\Delta + 1)^m + m$

Proof of the Steinitz Lemma

Background

- The original Steinitz Lemma with a worse bound was proven and published by Ernst Steinitz in 1913 (not motivated by ILPs)
- The currently best bound was proven by Sergey Sevastyanov in the 1970s
- There are several other major results in mathematics that also have the name **Steinitz Lemma**

Ernst Steinitz

Construction of permutation

We iteratively determine $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$ in that order

Determining $\sigma(n)$: Consider the linear program

$$\sum_{i \in \{1,\ldots,n\}} (v_i)_j x_i = 0 \qquad \text{for all } j \in \{1, \ldots, m\}$$

$$\sum_{i \in \{1,\ldots,n\}} x_i = n - 1 - m$$

$$x_i \in [0, 1] \qquad \text{for all } i \in \{1, 2, \ldots, n\}$$

- It is feasible because $x_1 = \cdots = x_n = (n - 1 - m)/n$ is a solution

Construction of permutation

We iteratively determine $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$ in that order

Determining $\sigma(n)$: Consider the linear program

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$$\begin{aligned} \sum_{i \in \mathbf{U_{n-1}}} (v_i)_j x_i &= 0 && \text{for all } j \in \{1, \ldots, m\} \\ \sum_{i \in \mathbf{U_{n-1}}} x_i &= |\mathbf{U_{n-1}}| - m && \\ x_i &\in [0,1] && \text{for all } i \in \mathbf{U_{n-1}} \end{aligned}$$

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Construction of permutation (cont.)

Determining $\sigma(n-1)$: Consider the linear program

$$\begin{aligned}
\sum_{i \in U_{n-1}} (v_i)_j x_i &= 0 && \text{for all } j \in \{1, \ldots, m\} \\
\sum_{i \in U_{n-1}} x_i &= |U_{n-1}| - m && \\
x_i &\in [0, 1] && \text{for all } i \in U_{n-1}
\end{aligned}$$

Construction of permutation (cont.)

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- If we decrease the right-hand side of the second constraint, then by scaling down solution, it remains feasible

Construction of permutation (cont.)

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Construction of permutation (cont.)

Determining $\sigma(n-1)$: Consider the linear program

$$\begin{aligned} \sum_{i \in \mathbf{U_{n-2}}} (v_i)_j x_i &= 0 && \text{for all } j \in \{1, \ldots, m\} \\ \sum_{i \in \mathbf{U_{n-2}}} x_i &= |\mathbf{U_{n-2}}| - m \\ x_i &\in [0,1] && \text{for all } i \in \mathbf{U_{n-2}} \end{aligned}$$

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- Let $U_{n-2} = \{1, 2, \ldots, n\} \setminus \{\sigma(n), \sigma(n-1)\}$ be the unassigned vector indices. $(x_i^*)_{i \in U_{n-2}}$ is a solution to the LP restricted to U_{n-2}

Construction of permutation (cont..)

We make this into a general rule

Determining $\sigma(k)$, $k > m$: Consider the linear program and assume that by previous construction it is feasible.

$$\begin{aligned}
\sum_{i \in U_k} (v_i)_j x_i &= 0 && \text{for all } j \in \{1, \ldots, m\} \\
\sum_{i \in U_k} x_i &= |U_k| - m && \\
x_i &\in [0, 1] && \text{for all } i \in U_k
\end{aligned}$$

Construction of permutation (cont..)

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- By previous Lemma it has a solution x^* with $\leq m + 1$ fractional variables
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Construction of permutation (cont..)

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$$\sum_{i \in \mathbf{U_{k-1}}} x_i = |\mathbf{U_{k-1}}| - m$$

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Construction of permutation (cont..)

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Determining $\sigma(k)$, $k \leq m$: once only $|U_m| = m$ vectors are left, assign them to $\sigma(m), \dots, \sigma(1)$ arbitrarily

Analysis: blackboard